## BIVARIATE RANDOM VECTOR

2 DECKS OF CARDS (4 CARDS IN EACH) NUMBERED 1, 2, 3, 4

$D_1$ : NUMBER DRAWN FROM DECK 1

$D_2$ : NUMBER DRAWN FROM DECK 2

2 CARDS ARE DRAWN – 1 FROM EACH GROUP.

$(x = D_1 ; \quad y = |D_1 - D_2)$ .

SAMPLE SPACE

| | $D_2 = 1$ | 2 | 3 | 4 |
|---|---|---|---|---|
| $D_1 = 1$ | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

MAPPING FROM SAMPLE SPACE ON TO $R^2$

$(x=1, y=0)$ $(x=1, y=1)$ $(x=1, y=2)$ $(x=1, y=3)$

$(x=2, y=1)$ $(x=2, y=0)$ $(x=2, y=1)$ $(x=2, y=2)$

$(x=3, y=2)$ $(x=3, y=1)$ $(x=3, y=0)$ $(x=3, y=1)$

$(x=4, y=3)$ $(x=4, y=2)$ $(x=4, y=1)$ $(x=4, y=0)$

# BIVARIATE PROBABILITY DISTRIBUTION

| | $y=0$ | 1 | 2 | 3 | $P(x)$ |
|---|---|---|---|---|---|
| $x=1$ | 1/16 | 1/16 | 1/16 | 1/16 | 1/4 |
| $=2$ | 1/16 | 2/16 | 1/16 | 0 | 1/4 |
| $=3$ | 1/16 | 2/16 | 1/16 | 0 | 1/4 |
| $=4$ | 1/16 | 1/16 | 1/16 | 1/16 | 1/4 |
| $P(y)$ | 1/4 | 6/16 | 1/4 | 1/8 | |

# MARGINAL PROBABILITY DISTRIBUTION

| $x=$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(x)$ | 1/4 | 1/4 | 1/4 | 1/4 |

$E(x) = 2\frac{1}{2}$

| $y=$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(y)$ | 1/4 | 3/8 | 1/4 | 1/8 |

$E(y) = 1\frac{1}{4}$

# CONDITIONAL PROBABILITY DISTRIBUTION

$$P(y=b \mid x=a) = \frac{P(y=b,\ x=a)}{P(x=a)}$$

| $y=$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(y \mid x=1)$ | $1/4$ | $1/4$ | $1/4$ | $1/4$ |

$E(y \mid x=1) = 1\frac{1}{2}$

| $y=$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(y \mid x=2)$ | $1/4$ | $1/2$ | $1/4$ | 0 |

$E(y \mid x=2) = 1$

| $y=$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(y \mid x=3)$ | $1/4$ | $1/2$ | $1/4$ | 0 |

$E(y \mid x=3) = 1$

| $y=$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(y \mid x=4)$ | $1/4$ | $1/4$ | $1/4$ | $1/4$ |

$E(y \mid x=4) = 1\frac{1}{2}$

NOTE THAT THE CONDITIONAL EXPECTATION $E(y|x)$ IS A FUNCTION OF $x$.

THIS IS THE <u>REGRESSION FUNCTION</u> OF $y$ ON $x$.

## BIVARIATE NORMAL DISTRIBUTION

2 INDEPENDENT RANDOM VARIABLES $u_1$ & $u_2$

$$f(u_1) = \frac{1}{\sqrt{2\pi}} e^{-u_1^2/2}$$

$$f(u_2) = \frac{1}{\sqrt{2\pi}} e^{-u_2^2/2}$$

THE JOINT PDF

$$h(u_1, u_2) = f(u_1) \cdot f(u_2)$$

$$= \frac{1}{2\pi} e^{-\left(\frac{u_1^2 + u_2^2}{2}\right)}$$

DEFINE $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

THEN $h(u) = \frac{1}{2\pi} e^{-\frac{u'u}{2}}$

NOW DEFINE

$$x_1 = a_{11} u_1 + a_{12} u_2 + b_1$$

$$x_2 = a_{21} u_1 + a_{22} u_2 + b_2$$

$$x = Au + b$$

$$E(x) = AE(u) + b = b$$

$$\Sigma = E\left[(x-b)(x-b)'\right]$$

$$x-b = Au$$

$$\Sigma = E\left[Auu'A'\right] = A\,E(uu')A'$$

$$E(uu') = I$$

$$\Rightarrow \quad \Sigma = AA'$$

$$u = A^{-1}(x-b)$$

$$u'u = (x-b)'A^{-1\prime}A^{-1}(x-b)$$

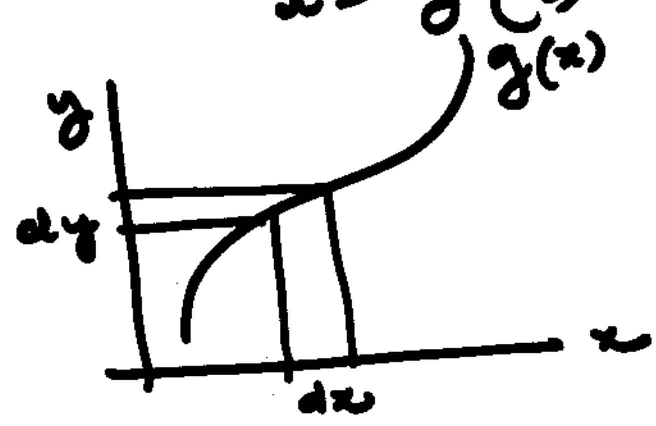
$$= (x-b)'(AA')^{-1}(x-b)$$

$$= (x-b)'\Sigma^{-1}(x-b)$$

## DIGRESSION

SUPPOSE $x$ HAS THE pdf $f(x)$ & $y = g(x)$ WHERE $g(\cdot)$ IS A MONOTONIC FUNCTION.

$$x = g^{-1}(y).$$




$$Pr(x \in dx) = Pr(y \in dy)$$

$$Pr(x \in dx) = f(x)\,dx$$

SUPPOSE $h(y)$ IS THE pdf OF $y$.

THEN $\Pr(y \in dy) = h(y)\,dy$

$$= f(x)\,dx = \Pr(x \in dx)$$

$$\Rightarrow \quad h(y) = f(x) \cdot \frac{dx}{dy}$$

$$= f(x)\,\frac{1}{dy/dx}$$

BUT $dy/dx$ MAY BE NEGATIVE. HENCE TO ENSURE THAT THE pdf $h(y)$ IS NON-NEGATIVE, WE TAKE $\left|dy/dx\right|$.

THUS $\quad h(y) = f(x) \cdot \left| \left(\frac{dy}{dx}\right)^{-1} \right|$.

EXAMPLE

$x \sim N(0,1)$. $\quad y = \sqrt{x} \quad \Rightarrow x = y^2$

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$h(y) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^4}{2}}\, |2y|$$

$$h(y) = \left(\sqrt{\frac{2}{\pi}}\right) e^{-\frac{y^2}{4}} |y|.$$

FOR THE BIVARIATE DISTRIBUTION

$x = Au + b$

THE JACOBIAN MATRIX IS

$$J = \left[\frac{\partial x}{\partial u}\right] = \begin{bmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 \end{bmatrix}$$

$$h(x_1, x_2) = f(u_1, u_2)\left[\,|J|\,\right]^{1}$$

IN OUR CASE $|J| = |A|$.

RECALL THAT $\Sigma = AA'$

$$|\Sigma| = |AA'| = (|A|)^2$$

$$|A| = |\Sigma|^{1/2}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \qquad |\Sigma| = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$$
$$= \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$$

$$h(x_1, x_2) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}(x-b)'\Sigma^{-1}(x-b)}$$

DEFINE

$$Q \equiv \frac{1}{2}(x-b)'\Sigma^{-1}(x-b).$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \qquad \Sigma^{-1} = \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} \Big/ |\Sigma|$$

$$= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \cdot \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix}$$

$$Q = \frac{1}{2\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} (x_1-b_1) & (x_2-b_2) \end{bmatrix} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} x_1-b_1 \\ x_2-b_2 \end{bmatrix}$$

$$= \frac{1}{2\sigma_2^2(1-\rho^2)} \left[ (x_2-b_2)^2 - 2\frac{\sigma_{12}}{\sigma_1^2}(x_1-b_1)(x_2-b_2) + (x_1-b_1)^2\frac{\sigma_2^2}{\sigma_1^2} \right]$$

$$Q = \frac{1}{2\sigma_2^2(1-\rho^2)} \left[ x_2 - \left(b_2 - \frac{\sigma_{12}}{\sigma_1^2}b_1\right) - \frac{\sigma_{12}}{\sigma_1^2}x_1 \right]^2 + \frac{1}{2\sigma_1^2}(x_1-b_1)^2.$$

4||  $(1-\rho^2)$

HENCE

$$h(x_1, x_2) = \left[\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x_1-b_1)^2}{2\sigma_1^2}}\right] \cdot \left[\frac{1}{\sqrt{2\pi(1-\rho^2)}\cdot\sigma_2}\right] e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left\{x_2 - \left(b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1\right) - \frac{\sigma_{12}}{\sigma_1^2} x_1\right\}^2}$$

THEN

$$f(x_2|x_1) = \frac{h(x_1, x_2)}{f(x_1)}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}\,\sigma_2} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[x_2 - \left(b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1\right) - \frac{\sigma_{12}}{\sigma_1^2} x_1\right]^2}$$

DEFINE $\sigma_2^2(1-\rho^2) = \hat{\sigma}_2^2$

& $b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1 + \frac{\sigma_{12}}{\sigma_1^2} x_1 = \hat{b}_2(x_1)$.

THEN $f(x_2|x_1) \sim N\left(\hat{b}_2(x_1), \hat{\sigma}_2^2\right)$.

THUS THE CONDITIONAL DISTRIBUTION OF $x_2$ GIVEN $x_1$ IS NORMAL WITH

$$E(x_2 | x_1) = \alpha + \beta x_1$$

$$\alpha = b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1$$

$$\beta = \frac{\sigma_{12}}{\sigma_1^2} .$$

NOTE: (1) THE CONDITIONAL MEAN OF $x_2$ GIVEN $x_1$ IS A FUNCTION OF $x_1$.

(2) THE CONDITIONAL VARIANCE OF $x_2$ GIVEN $x_1$ IS NOT A FUNCTION OF $x_2$. THE VARIANCE IS A CONSTANT. THIS PROPERTY IS KNOWN AS HOMOSCEDASTICITY.