MORE MATRIX ALGEBRA

CONSIDER THE EQUATIONS

$$a_{11} x_1 + a_{12} x_2 = \lambda x_1 \qquad (1)$$
$$a_{21} x_1 + a_{22} x_2 = \lambda x_2$$

IN MATRIX FORM,

$$Ax = \lambda x \qquad \text{WHERE } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

CLEARLY IF x^0 SATISFIES THIS EQUATION, SO DOES THE VECTOR tx^0 FOR ANY t.
SO WE NORMALIZE BY THE RESTRICTION

$$x'x = 1.$$

NOTE THAT (1) $\Rightarrow$

$$(a_{11} - \lambda) x_1 + a_{12} x_2 = 0$$
$$a_{21} x_1 + (a_{22} - \lambda) x_2 = 0$$

IN MATRIX FORM

$$Ax - \lambda x = (A - \lambda I) x = 0$$

FOR A NON-TRIVIAL SOLUTION (i.e., OTHER THAN $x = 0$) WE NEED THAT THE MATRIX $A - I\lambda$ MUST BE SINGULAR.

THUS

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0$$

THIS IS A QUADRATIC EQUATION IN λ WITH ROOTS λ_1 & λ_2.

THIS IS KNOWN AS THE <u>CHARACTERISTIC EQUATION</u> OF THE MATRIX A & THE ROOTS ARE ITS <u>CHARACTERISTIC ROOTS</u> OR <u>EIGEN VALUES</u>.

EXAMPLE:

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(4-\lambda) - 2 = 0$$

$$\lambda^2 - 9\lambda + 20 - 2 = 0 \Rightarrow \lambda^2 - 9\lambda + 18 = 0$$

$\lambda_1 = 6$, $\lambda_2 = 3$ ARE THE ROOTS.

CHARATERISTIC VECTORS

FOR $\lambda_1 = 6$, $Ax = \lambda_1 x$

$$\Rightarrow \left.\begin{array}{l} 5x_1 + x_2 = 6x_1 \\ 2x_1 + 4x_2 = 6x_2 \end{array}\right\} \Rightarrow x_1 = x_2$$

FURTHER, $x'x = 1 \Rightarrow x_1^2 + x_2^2 = 1 \Rightarrow 2x_1^2 = 1 \Rightarrow x_1 = x_2 = 1/\sqrt{2}$

THE CHARACTERTIC VECTOR ASSOCIATED WITH $\lambda_1 = 6$ IS

$$x^1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

FOR $\lambda_2 = 3$,

$$\left.\begin{array}{l} 5x_1 + x_2 = 3x_1 \\ 2x_1 + 4x_2 = 3x_2 \end{array}\right\} \Rightarrow \begin{array}{l} x_2 = -2x_1 \\ x_2^2 = 4x_1^2 \end{array}$$

$$x'x = 1 \Rightarrow x_1^2 + x_2^2 = 1 \Rightarrow 5x_1^2 = 1. \qquad x_1 = 1/\sqrt{5}, \quad x_2 = -2/\sqrt{5} \ldots$$

FOR $\lambda_2 = 3$, THE CORRESPONDING CHARACTERISTIC ROOT IS $x^2 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

ASSUME THAT A IS A SYMMETRIC MATRIX OF ORDER K. THEN A HAS K CHARACTERISTIC ROOTS (MAY NOT BE ALL DIFFERENT).

THE CHARECTERISTIC VECTORS SATISFY

$$Ax^1 = \lambda_1 x^1; \quad Ax^2 = \lambda_2 x^2, \ldots, Ax^k = \lambda_k x^k.$$

THEN
$$x^{1\prime}Ax^1 = x^{1\prime}(\lambda_1 x^1) = \lambda_1 x^{1\prime}x^1 = \lambda_1$$
$$x^{2\prime}Ax^2 = x^{2\prime}(\lambda_2 x^2) = \lambda_2 x^{2\prime}x^2 = \lambda_2$$
$$\vdots$$
$$x^{k\prime}Ax^k = x^{k\prime}(\lambda_k x^k) = \lambda_k x^{k\prime}x^k = \lambda_k$$

NOW CONSIDER 2 DIFFERENT VECTORS x^i & x^j.

$$Ax^i = \lambda_i x^i$$
$$x^{j\prime}Ax^i = \lambda_i x^{j\prime}x^i$$

FURTHER,
$$Ax^j = \lambda_j x^j$$
$$x^{i\prime}Ax^j = \lambda_j x^{i\prime}x^j$$

BY SYMMETRY OF A,
$$[x^{j\prime}Ax^i]' = x^{i\prime}A'x^j = x^{i\prime}Ax^j$$

THUS
$$[\lambda_j x^{j\prime}x^i]' = \lambda_j x^{i\prime}x^j$$
$$= x^{i\prime}Ax^j = \lambda_i x^{i\prime}x^j$$

THIS IS TRUE FOR ANY PAIR (λ_i, λ_j) NOT NECESSARILY EQUAL TO ONE ANOTHER.

HENCE $\lambda_i x^{i\prime}x^j = \lambda_j x^{i\prime}x^j \Rightarrow x^{i\prime}x^j = 0$ FOR ANY PAIR $i \neq j$

THE CHARACTERISTIC VECTORS ARE PAIRWISE ORTHOGONAL.

DIAGONALIZATION OF MATRICES

CONSIDER THE MATRIX
$$X = [x^1 \mid x^2 \mid \cdots \mid x^k]$$
& THE DIAGONAL MATRIX
$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix}$$

THEN
$$AX = A(x^1 \mid x^2 \mid \cdots \mid x^k)$$
$$= (Ax^1 \mid Ax^2 \mid \cdots \mid Ax^k) = (\lambda_1 x^1 \mid \lambda_2 x^2 \mid \cdots \mid \lambda_k x^k)$$
$$\Rightarrow AX = X\Lambda$$
$$X'AX = X'X\Lambda = \Lambda \quad (\text{RECALL THAT } X'X = I).$$

THUS $\boxed{X'AX = \Lambda}$

EXAMPLE: $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$; $|A - \lambda I| = 0$

$$\begin{vmatrix} (5-\lambda) & 2 \\ 2 & (5-\lambda) \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 10\lambda + 25 - 4 = 0$$

$$\lambda^2 - 10\lambda + 21 = 0 \Rightarrow \lambda_1 = 7, \quad \lambda_2 = 3$$

$$Ax^1 = \lambda_1 x^1 \Rightarrow \begin{aligned} 5x_1 + 2x_2 &= 7x_1 \\ 2x_1 + 5x_2 &= 7x_2 \end{aligned} \quad \Bigg| \quad x^1 = \begin{pmatrix} 1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix}$$

$$Ax^2 = \lambda_2 x^2 \Rightarrow \begin{aligned} 5x_1 + 2x_2 &= 3x_1 \\ 2x_1 + 5x_2 &= 3x_2 \end{aligned} \quad \Bigg| \quad x^2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$X = (x^1 \vdots x^2) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad \text{VERIFY, } X'X = I$$

$$X'AX = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 7/\sqrt{2} & 3/\sqrt{2} \\ 7/\sqrt{2} & -3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$$

CHARACTERISTIC ROOTS OF DEFINITE MATRICES

~~ARE~~ THE MATRIX OF CHARACTERISTIC VECTORS OF A IS X SATISFYING $X'X = I$

THUS $X' = X^{-1}$ & $XX' = XX^{-1} = I$

RECALL, $X'AX = \Lambda$

$$\Rightarrow XX'AXX' = X\Lambda X'$$

$$\Rightarrow A = X\Lambda X'$$

HENCE, $y'Ay = y'X\Lambda X'y$

DEFINE $X'y \equiv z$. THEN, $z'\Lambda z = y'Ay$

$$= \sum \lambda_j z_j^2 \geq 0$$

FOR ANY y IF $\lambda_j \geq 0$ FOR ALL j.

BUT $y'Ay \geq 0 \Rightarrow A$ IS POSITIVE SEMI-DEFINITE.

DETERMINANT OF A SYMMETRIC MATRIX A

RECALL THAT $A = X\Lambda X'$

THUS $|A| = |X\Lambda X'| = |X|\cdot|\Lambda|\cdot|X'| = |X'|\cdot|X|\cdot|\Lambda|$

NOW, $|X'X| = |I| = 1$

ALSO $|X'X| = |X'|\cdot|X|$

HENCE, $|A| = |X'X|\cdot|\Lambda| = |\Lambda| = \prod_j \lambda_j$.

IF ANY ONE OF THE CHARACTERISTIC ROOTS OF A EQUALS 0, A IS SINGULAR.

TRACE OF A SYMMETRIC MATRIX

$$A = X\Lambda X'$$

$$tr(A) = tr(X\Lambda X') = tr(\Lambda X'X) = tr(\Lambda).$$

$$= \sum_j \lambda_j .$$

TRACE OF A EQUALS THE <u>SUM</u> OF ITS CHARACTERISTIC ROOTS.

CHARACTERISTIC ROOTS OF A SYMMETRIC IDEMPOTENT MATRIX

LET A BE A SYMMETRIC & IDEMPOTENT MATRIX. THEN $A' = A$ & $A'A = A.A = A$

LET λ_j BE THE jth ROOT & x^j THE ASSOCIATED CHARACTERISTIC VECTOR. THEN

$$Ax^j = \lambda_j x^j$$

HENCE $A(Ax^j) = A(\lambda_j x^j) = \lambda_j (Ax^j) = \lambda_j(\lambda_j x^j) = \lambda^2 x^j$

THUS $\lambda_j x^j = \lambda_j^2 x^j \Rightarrow (\lambda_j - \lambda_j^2)\, x^j = \underline{0}$

HENCE $\lambda_j - \lambda_j^2 = 0 \Rightarrow \lambda_j = 0, 1$.

THUS ALL CHARACTERISTIC ROOTS OF A SYMMETRIC IDEMPOTENT MATRIX MUST EQUAL EITHER 0 OR 1.

RANK OF A SYMMETRIC IDEMPOTENT MATRIX A

$$A = X \Lambda X'$$

$$\text{RANK}(A) = \text{RANK}(X \Lambda X')$$
$$= \min\left[\text{RANK}(X),\ \text{RANK}(\Lambda)\right]$$

NOTE $\quad X'X = I$

$$\text{RANK}(X'X) = \text{RANK}(I) = n$$

BUT $\quad \text{RANK}(X'X) = \text{RANK}(X) \quad \Rightarrow \text{RANK}(X) = n$

HENCE $\quad \text{RANK}(A) = \min(n,\ \text{RANK}(\Lambda))$.

NOW, $\quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

HENCE, $\quad \text{RANK}(\Lambda) = \#$ OF NON-ZERO λ_js.

THIS IS TRUE FOR ALL SYMMETRIC MATRICES. BUT IF A IS ALSO IDEMPOTENT, EACH λ_j IS EITHER 0 OR 1. IN THIS CASE, THE NUMBER OF NON-ZERO CHARACTERISTIC ROOTS $= \sum \lambda_j = tr(\Lambda)$.

BUT $\quad tr(\Lambda) = tr(A)$.

HENCE THE RANK OF $A = tr(A)$.

DISTRIBUTION OF A QUADRATIC FORM

CONSIDER A RANDOM VECTOR $u \sim N(0, I)$.

LET $\quad q = u'Au \quad$ WHERE A IS SYMMETRIC.

THEN $\quad A = X \Lambda X'$

& $\quad q = (u'X)\Lambda(X'u)$

DEFINE $\quad y = X'u. \quad$ THEN $E(y) = X'E(u) = 0$

$$V(y) = E[yy'] = E[X'uu'X] = X'E(uu')X = X'X = I$$

THUS $\quad y \sim N(0, I)$

HENCE, $q = y'\Lambda y = \sum_j \lambda_j y_j^2$ IS A LINEAR COMBINATION OF SQUARES ON STANDARD NORMAL VARIABLES (y_j).

FURTHER, IF A IS SYMMETRIC & IDEMPOTENT, EACH λ_j IS EITHER 0 OR 1. SUPPOSE THAT p IS THE NUMBER OF NON-ZERO CHARACTERISTIC ROOTS. THEN q IS THE SUM OF SQUARES OF p RANDOM VARIABLES y_j EACH $\sim N(0,1)$. [NOTE THAT p IS THE TRACE OF A AS SHOWN BEFORE.]

THEREFORE $q \sim \chi^2_p$.

INDEPENDENCE OF QUADRATIC FORMS

CONSIDER A RANDOM VECTOR x WITH

$$E(x) = \underline{0} \quad \& \quad V(x) = E[xx'] = I$$

& TWO QUADRATIC FORMS

$$q_1 = x'Ax \quad \& \quad q_2 = x'Bx.$$

ASSUME THAT BOTH A & B ARE IDEMPOTENT MATRICES (ALSO SYMMETRIC).

$$\text{THEN } q_1 = x'A'Ax \quad \& \quad q_2 = x'B'Bx$$

DEFINE $x_1 = Ax$ & $x_2 = Bx$

THEN $q_1 = x_1'x_1$ & $q_2 = x_2'x_2$

BECAUSE $E(x)=0$, $E(x_1) = AE(x) = 0$ &

$$E(x_2) = BE(x) = 0.$$

$$E[x_1x_2'] = E[Axx'B'] = AE(xx')B' = AB' = AB.$$

THUS x_1 & x_2 ARE INDEPENDENT (AND HENCE q_1 & q_2 ARE INDEPENDENT) IN $AB=0$.

NOW CONSIDER

$$x'x = \Sigma x_i^2 = \Sigma (x_i - \bar{x})^2 + n\bar{x}^2$$

DEFINE $\boxed{q_1 = \Sigma (x_i - \bar{x})^2 = x' M_0 x}$

WHERE $M_0 = I - \frac{1}{n} i i'$ [AS SHOWN BEFORE).

BECAUSE $x \sim N(0, I)$, $q_1 \sim \chi^2_p$

WHERE $p = tr(M_0)$.

BUT $tr(M_0) = tr\left[I - \frac{1}{n} ii'\right] = tr[I] - \frac{1}{n} tr[ii']$

$$= n - \frac{1}{n} \cdot n = n - 1.$$

THUS $q_1 \sim \chi^2_{n-1}$.

NEXT DEFINE $q_2 = n\bar{x}^2 = n \cdot \frac{(\Sigma x_i)^2}{n^2}$

BUT $\Sigma x_i = i'x$. THUS $q_2 = n \frac{[x'ii'x]}{n^2}$

DEFINE THE VECTOR $j = \frac{1}{\sqrt{n}} i$

THEN $jj' = \frac{ii'}{n}$ & $q_2 = x'[jj']x$

DEFINE $P^0 = I - M_0 = I - \left(I - \frac{1}{n} ii'\right) = \frac{1}{n} ii' = [jj']$

THEN $\boxed{q_2 = x'P^0 x}$

$M_0 P_0 = M_0 (I - M_0) = M_0 - M_0 . M_0 = M_0 - M_0 = 0$.

THUS q_1 & q_2 ARE INDEPENDENT.

$tr(P_0) = tr(I - M_0) = tr(I) - tr(M_0) = n - (n-1) = 1$

HENCE $q_1 = \Sigma (x_i - \bar{x})^2 \sim \chi^2_{n-1}$

& $q_2 = n\bar{x}^2 \sim \chi^2_1$

WHERE q_1 & q_2 ARE INDEPENDENT.

INDEPENDENCE OF A LINEAR & A QUADRATIC FORM

CONSIDER A QUADRATIC FORM

$$q = x'Ax$$

& A VECTOR $p = Lx$

HERE A IS SYMMETRIC IDEMPOTENT MATRIX

& L IS A $k \times n$ MATRIX.

WE WANT TO KNOW UNDER WHAT CONDITION WILL BE THE SINGLE RANDOM VARIABLE q INDEPENDENT OF EACH ELEMENT p_j $(j = 1, 2, \cdots, n)$ OF THE VECTOR p.

BECAUSE A IS A SYMMETRIC IDEMPOTENT MATRIX,

$$x'Ax = x'A'Ax.$$

DEFINE $r = Ax$. THEN

$$q = x'Ax = r'r.$$

NOTE THAT $E(p) = L\,E(x) = 0$

& $E(r) = A\,E(x) = 0$.

THE COVARIANCE MATRIX BETWEEN p & r IS

$$E(pr') = E[Lxx'A'] = L\,E(xx')\,A'$$

BUT $x \sim N(0, I)$ & $E(xx') = I$

HENCE $E(pr') = LA' = LA$

IF ELEMENTS OF THE VECTORS p & r ARE INDEPENDENT, $E(pr') = 0 \Rightarrow LA = 0$.

OF COURSE WHEN p & r ARE INDEPENDENT, BECAUSE $q = r'r$, p & q ARE ALSO INDEPENDENT.

THE OLS COEFFICIENT VECTOR:

$$\hat{\beta} = \beta + (X'X)^{-1}X'u$$

$$E(\hat{\beta}) = \beta$$

$$\& \quad V(\hat{\beta}) = \sigma^2(X'X)^{-1}.$$

IF $\quad u \sim N(0, \sigma^2 I), \quad \hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1}).$

CONSIDER THE QUADRATIC FORM

$$\frac{e'e}{\sigma^2} = \frac{u'Mu}{\sigma^2} = \left(\frac{u}{\sigma}\right)' M \left(\frac{u}{\sigma}\right).$$

BECAUSE $\quad u \sim N(0, \sigma^2 I), \quad \left(\frac{u}{\sigma}\right) \sim N(0, I).$

HENCE, $\quad \frac{e'e}{\sigma^2} \sim \chi^2_{n-k}.$

NOW CONSIDER THE LINEAR FORM

$$\frac{\hat{\beta} - \beta}{\sigma} = \frac{(X'X)^{-1}X'u}{\sigma} = (X'X)^{-1}X'\left(\frac{u}{\sigma}\right)$$

DEFINE $\quad (X'X)^{-1}X' \equiv A$

THEN $\quad \frac{\hat{\beta} - \beta}{\sigma} = A\left(\frac{u}{\sigma}\right)$

NOW
$$AM = (X'X)^{-1}X'\left[I - X(X'X)^{-1}X'\right]$$
$$= (X'X)^{-1}X' - (X'X)^{-1}X'X(X'X)^{-1}X'$$
$$= (X'X)^{-1}X' - (X'X)^{-1}X' = \underline{0}$$

THEREFORE $\quad \frac{\hat{\beta} - \beta}{\sigma} \;\&\; \frac{e'e}{\sigma^2}$ ARE

INDEPENDENT

ANY INDIVIDUAL ELEMENT OF $\frac{\hat{\beta}-\beta}{\sigma}$, SAY $\frac{\hat{\beta}_k-\beta_k}{\sigma}$ HAS THE VARIANCE s^{kk} WHERE s^{kk} IS THE kth DIAGONAL ELEMENT OF $(X'X)^{-1}$.

HENCE $$\left(\frac{\hat{\beta}_k-\beta_k}{\sigma}\right)\frac{1}{\sqrt{s^{kk}}} \sim N(0,1)$$

& $$\frac{(\hat{\beta}_k-\beta_k)^2}{\sigma^2(s^{kk})} \sim \chi^2_1 .$$

FURTHER $\frac{e'e}{\sigma^2} \sim \chi^2_{n-k}$ & IS INDEPENDENT OF $\frac{\hat{\beta}_k-\beta_k}{\sigma\sqrt{s^{kk}}}$. THUS THE RATIO

$$\frac{\dfrac{\hat{\beta}_k-\beta_k}{\sigma\sqrt{s^{kk}}}}{\sqrt{\dfrac{e'e}{\sigma^2(n-k)}}} = \frac{\hat{\beta}_k-\beta_k}{\sqrt{s^{kk}\;\hat{\sigma}^2}} \sim t_{n-k}$$

WHERE $\hat{\sigma}^2 = \frac{e'e}{n-k}$.

IN THE 2- VARIABLE REGRESSION

$$(X'X)^{-1} = \frac{1}{n\sum x_i^2-(\sum x_i)^2}\begin{bmatrix}\sum x_i^2 & -\sum x_i \\ -\sum x_i & n\end{bmatrix}$$

THUS $$s^{22} = \frac{1}{\sum x_i^2 - n\bar{x}^2} = \frac{1}{\sum(x_i-\bar{x})^2}$$

THUS $$t = \frac{\hat{\beta}_2-\beta_2}{\sqrt{\hat{\sigma}^2/\sum(x_i-\bar{x})^2}} \sim t_{n-2}$$