

POINT & INTERVAL ESTIMATES

OLS ESTIMATOR $\hat{\beta} = (X'X)^{-1}X'y$ USES SAMPLE INFORMATION TO PROVIDE SPECIFIC NUMERICAL ESTIMATES OF THE COEFFICIENTS β_j . AN INTERVAL ESTIMATOR USES THE SAMPLING VARIANCE OF THE ESTIMATOR TO CONSTRUCT AN INTERVAL AROUND THE SPECIFIC VALUES OF THE COEFFICIENTS, WHICH ARE THE POINT ESTIMATES.

TYPE I & TYPE II ERRORS

A HYPOTHESIS H_0 IS EITHER TRUE OR FALSE. BUT BASED ON SAMPLE INFORMATION ONLY ONE CANNOT KNOW THE TRUE STATE OF THE WORLD & MUST MAKE A DECISION ABOUT TRUTH/FALSEHOOD OF H_0 BASED ON LIMITED INFORMATION ONLY.

STATE OF THE WORLD	DECISIONS	
	ACCEPT	REJECT
H_0 IS TRUE	CORRECT	TYPE I ERROR
H_0 IS FALSE	TYPE II ERROR	CORRECT

GENERALLY TYPE I ERROR IS TREATED AS MORE SERIOUS. THE DECISION RULE IS BASED ON A CONFIDENCE INTERVAL SO CONSTRUCTED THAT THE PROBABILITY THAT THE ~~SAMPLE~~ ESTIMATOR WOULD GENERATED AN ESTIMATE OUTSIDE OF THE INTERVAL WHEN H_0 IS TRUE IS ONLY α . HENCE IF H_0 IS REJECTED WHENEVER THE ESTIMATE FALLS OUTSIDE THE INTERVAL, THE PROBABILITY OF TYPE I ERROR IS α .

FROM A t -DISTRIBUTION WITH $p = n - k$ DEGREES OF FREEDOM, WE CAN FIND THE VALUE OF THE t -STATISTIC $t_{\alpha/2}^*$ SUCH THAT

$$PR \{ t \geq t_{\alpha/2}^* \} = \alpha/2$$

BY SYMMETRY,

$$PR \{ t \leq -t_{\alpha/2}^* \} = \alpha/2$$

$$\text{HENCE } PR \{ -t_{\alpha/2}^* \leq t \leq t_{\alpha/2}^* \} = 1 - \alpha.$$

HERE $(1 - \alpha)$ IS THE CONFIDENCE LEVEL & α IS THE SIGNIFICANCE LEVEL.

FOR THE 2-VARIABLE REGRESSION

$$t = \frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)} = \frac{\hat{\beta}_2 - \beta_2}{\frac{\hat{\sigma}}{\sqrt{\sum (x_i - \bar{x})^2}}} \sim t_{n-2}$$

SUPPOSE THAT FOR $n-2$ d.f. $PR \{ t \geq t^* \} = \alpha/2$

THEN

$$PR \left\{ -t^* \leq \frac{\hat{\beta}_2 - \beta_2}{se(\hat{\beta}_2)} \leq t^* \right\} = 1 - \alpha$$

$$\Rightarrow PR \left\{ -t^* se(\hat{\beta}_2) \leq \hat{\beta}_2 - \beta_2 \leq t^* se(\hat{\beta}_2) \right\} = 1 - \alpha$$

$$\Rightarrow PR \left\{ \hat{\beta}_2 - t^* se(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t^* se(\hat{\beta}_2) \right\} = 1 - \alpha$$

THE INTERVAL $\left(\hat{\beta}_2 \pm t^* se(\hat{\beta}_2) \right)$ IS AN INTERVAL ESTIMATOR OF β_2 . $\hat{\beta}_2$ IS A POINT ESTIMATOR OF β_2 .

TEST OF HYPOTHESIS $H_0: \beta_2 = \beta_2^0$
AGAINST $H_1: \beta_2 \neq \beta_2^0$

$$t = \frac{\hat{\beta}_2 - \beta_2^0}{se(\hat{\beta}_2)}$$

REJECT H_0 IF $|t| > t^*$.

DO NOT REJECT IF $|t| < t^*$.

EXAMPLE

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$\bar{Y} = 20; \quad \sum (X_i - \bar{X})^2 = 60;$$
$$\bar{X} = 10; \quad \sum (Y_i - \bar{Y})^2 = 100;$$

$$\sum (X_i - \bar{X})(Y_i - \bar{Y}) = 30; \quad n = 22$$

$$\hat{\beta}_2 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{30}{60} = 0.5$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 20 - (0.5)(10) = 15$$

$$SST = \sum (Y_i - \bar{Y})^2 = 100$$

$$SSR = (\hat{\beta}_2)^2 \sum (X_i - \bar{X})^2 = 15$$

$$SSE = \sum e_i^2 = SST - SSR = 85$$

$$\hat{\sigma}^2 = \frac{\sum e_i^2}{n-2} = \frac{85}{22-2} = 4.25$$

$$VAR(\hat{\beta}_2) = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2} = \frac{4.25}{60} = 0.0708$$

$$SE(\hat{\beta}_2) = \sqrt{VAR(\hat{\beta}_2)} = 0.2661$$

SIGNIFICANCE LEVEL = $\alpha = 0.05$

$$t_{\alpha/2} = 2.086 = (t^*) \quad (n-2 = 20)$$

HYPOTHESIS TESTING:

$$H_0: \beta_2 = 1.0 \quad (= \beta_2^0)$$

$$t = \frac{\hat{\beta}_2 - \beta_2^0}{SE(\hat{\beta}_2)} = \frac{0.05 - 1.0}{0.2661}$$

$$= -1.879$$

$$|t| = 1.879 < 2.089 (= t^*)$$

TEST OF SIGNIFICANCE OF OVERALL REGRESSION

CONSIDER THE K VARIABLE REGRESSION

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \dots + \beta_k x_{ki} + u_i$$

AND $H_0: \beta_2 = \beta_3 = \dots = \beta_k = 0$

IF H_0 IS TRUE $y_i = \beta_1 + u_i$, $\hat{\beta}_1 = \bar{y}$

$$e_i = y_i - \hat{\beta}_1 = y_i - \bar{y}$$

$$\sum e_i^2 = \sum (y_i - \bar{y})^2 \Rightarrow SSE = SST.$$

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} = 0.$$

WE WANT TO TEST IF R^2 IS SIGNIFICANTLY DIFFERENT FROM 0.

THE RATIO

$$\frac{R^2}{1-R^2} = \frac{SSR/SST}{SSE/SST} = \frac{SSR}{SSE} = \frac{\hat{\beta}' X' M_0 X \hat{\beta}}{e'e}$$

CONSIDER THE MATRIX

$$X = \begin{bmatrix} 1 & x_{12} & \dots & x_{1k} \\ 1 & x_{22} & & x_{2k} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n2} & & x_{nk} \end{bmatrix}$$

& A MATRIX

$$J = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

X IS $n \times k$ & J IS A $k \times (k-1)$ MATRIX.

THEN
$$XJ = \begin{bmatrix} 1 & x_{12} & \dots & x_{1k} \\ 1 & x_{22} & & x_{2k} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n2} & & x_{nk} \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} x_{12} & x_{1k} \\ x_{22} & x_{2k} \\ \vdots & \vdots \\ x_{n2} & x_{nk} \end{bmatrix}$$

PARTITION

$$\beta \text{ AS } \beta = \begin{pmatrix} \beta_1 \\ \delta \end{pmatrix}$$

WHERE

$$\delta = \begin{pmatrix} \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \end{pmatrix}$$

6-5

THEN

$$y = \beta_1 + x'J\delta + u$$

$$M_0 y = M_0 x'J\delta + M_0 u$$

DEFINE

$$M_0 x'J \equiv Z$$

THEN

$$M_0 y = Z\delta + M_0 u$$

$$\hat{\delta} = (Z'Z)^{-1} Z' M_0 y$$

$$= (Z'Z)^{-1} Z' M_0 (Z\delta + M_0 u)$$

$$= (Z'Z)^{-1} (Z' M_0 Z) \delta + (Z'Z)^{-1} Z' M_0 u$$

NOW

$$Z = M_0 x'J$$

$$Z' = J' x' M_0$$

$$Z' M_0 = J' x' M_0 M_0 = J' x' M_0 = Z'$$

THUS

$$Z' M_0 Z = Z'Z \quad \& \quad Z' M_0 u = Z'u$$

HENCE

$$\hat{\delta} = \delta + (Z'Z)^{-1} Z'u$$

IF

$$\delta = 0, \quad \hat{\delta} = (Z'Z)^{-1} Z'u$$

HENCE

$$\hat{\delta}' Z' Z \hat{\delta} = u' Z (Z'Z)^{-1} Z' Z (Z'Z)^{-1} Z'u$$

$$= u' Z (Z'Z)^{-1} Z'u$$

DEFINE

$$P_Z = Z(Z'Z)^{-1} Z' \quad \& \quad M_Z = I - P_Z$$

CLEARLY

BOTH

MATRICES ARE SYMMETRIC &

IDEMPOTENT.

ALSO

$$e = u' M u$$

HENCE,

$$\frac{R^2}{1-R^2} = \frac{\frac{\hat{\delta}' Z Z \hat{\delta}}{\sigma^2}}{\frac{e'e}{\sigma^2}} = \frac{\left(\frac{u}{\sigma}\right)' P_Z \left(\frac{u}{\sigma}\right)}{\left(\frac{u}{\sigma}\right)' M \left(\frac{u}{\sigma}\right)}$$

FINALLY

$$P_Z M = (I - MZ) M$$

$$= Z(Z'Z)^{-1} Z' M$$

NOW

$$Z = M_0 X J$$

$$Z' = J' X' M_0$$

HENCE

$$Z' M = J' X' M_0 M$$

$$= J' X' \left(I - \frac{ZZ'}{n} \right) M$$

$$= J' X' M - J' X' \left(\frac{ZZ'}{n} \right) M$$

BUT

$$X' M = 0 \Rightarrow Z' M = 0$$

$$\text{HENCE } Z' M = 0 \Rightarrow$$

$$P_Z M = 0$$

HENCE

$$\left(\frac{u}{\sigma}\right)' P_Z \left(\frac{u}{\sigma}\right) \quad \&$$

$$\left(\frac{u}{\sigma}\right)' M \left(\frac{u}{\sigma}\right) \quad \text{ARE}$$

INDEPENDENTLY DISTRIBUTED

THUS

$$\frac{\hat{\delta}' Z Z \hat{\delta}}{\sigma^2} \sim \chi^2_{R-1} \quad \&$$

$$\frac{e'e}{\sigma^2} \sim \chi^2_{n-k}$$

$\frac{R^2/R-1}{1-R^2/n-R} = F$ IS A RATIO OF INDEPENDENT χ^2 VARIATES EACH DIVIDED BY THE RELEVANT DEGREES OF FREEDOM AND HAS THE χ^2 DISTRIBUTION.