Answer the questions fully to your best ability. Use the space provided. If you run out of room, use the backsides. No partial credit will be given if you do not show the steps of your calculations! **Write as neatly as possible!**

Name: ____________________

1. (30min) A firm uses capital K and labor L to produce units of output (Q). The per-unit prices of capital and labor are r and w, respectively, and the firm receives a price of p per unit of output. Government is concerned about the level of unemployment and wants to encourage the firm to hire more labor. To do this, it offers the firm a labor subsidy (s), where $0 < s < 1$. The subsidy parameter, s, is the fraction of the market wage paid by the government, so the effective price of labor for the firm is $(1-s)w$. If the production function of the firm is:

$$Q(K, L) = a\ln(K) + b\ln(L),$$

where $a, b > 0$, then the firm's profit function is:

$$\Pi[K, L; s] = p[a\ln(K) + b\ln(L)] - rK - (1-s)wL$$

(5) (a) Specify the first order condition for profit-maximization.

Solution:

$$\Pi_K = \frac{pa}{K} - r = 0$$

$$\Pi_L = \frac{pb}{L} - (1-s)w = 0$$

(5) (b) Solve the first-order conditions for the profit-maximizing input demands, K^* and L^*, each expressed as a function of the parameters of the problem (a, b, p, r, w, s). [Note: not all parameters may appear in each expression.]

Solution:

$$\Pi_K = \frac{pa}{K} - r = 0 \Rightarrow K^* = \frac{pa}{r}$$

$$\Pi_L = \frac{pb}{L} - (1-s)w = 0 \Rightarrow L^* = \frac{pb}{(1-s)w}$$

(5) (c) Using results from (b), find the firm's supply function, $Q^*(a, b, p, r, w, s)$.

Solution:

$$Q^* = a\ln\left(\frac{pa}{r}\right) + b\ln\left(\frac{pb}{(1-s)w}\right)$$

Question 1 continues on the next page...

(5) (d) Estimate the effect of a change in subsidy rate s on K^* and L^*. [Hint: Find the Jacobian of the first-order conditions and apply Cramer's Rule.]

Solution:

$$\frac{\partial K^*}{\partial s} = \frac{\begin{vmatrix} 0 & 0 \\ -w & -pbL^{-2} \end{vmatrix}}{\begin{vmatrix} -paK^{-2} & 0 \\ 0 & -pbL^{-2} \end{vmatrix}} = 0$$

$$\frac{\partial L^*}{\partial s} = \frac{\begin{vmatrix} -paK^{-2} & 0 \\ 0 & -w \end{vmatrix}}{\begin{vmatrix} -paK^{-2} & 0 \\ 0 & -pbL^{-2} \end{vmatrix}} = \frac{wL^2}{pb}$$

(5) (e) Differentiate the K^* and L^* expressions you got in part (b) with respect to s. Is your answer different from what you got in part (d)?

Solution: $\partial K^*/\partial s = 0$ and $\partial L^*/\partial s = \frac{bp}{w(1-s)^2}$.
It looks that we have a difference answer in part (b) for $\partial L^*/\partial s$. But if we substitute L^* into that result, we would get the same answer.

2. (30 min) The Connecticut Department of Transportation (DOT) plans to construct a highway between two given points. It must decide on the number of lanes to be built (n) and the width (w) of each lane. The flow (f) of traffic that each lane can support depends on the lane width, or: $f(w)$, where $f_w > 0$ and $f_{ww} < 0$. The total flow capacity of the road is the number of lanes times the flow per lane, or $n \cdot f(w)$. In addition to a fixed cost of initiating any road construction project, costs increase linearly with the total width ($n \cdot w$)of the roads. Thus, total costs of the project are given by: $anw + c$, where $a > 0$ is the cost per unit of road width and $c > 0$ is the fixed cost. The DOT has a fixed budget (B) for this project and wants total costs to equal this budgeted amount.

(10) (a) Suppose DOT wants to select n and w so as to maximize total flow capacity, $n \cdot f(w)$, subject to the equality constraint that $B = anw + c$. Ignoring the fact that, in reality, n would have to be an integer, give the constrained maximization problem in Lagrangian form.

Solution: $\max_{n,w} L = nf(w) + \lambda(B - anw - c)$

(5) (b) Derive the first-order conditions for a maximum. What do these conditions imply about the value of the expression $(f_w - \lambda a)$ at the optimum? What will be the economic interpretation of λ^* in this particular problem?

Solution:

$$\begin{aligned} L_n = f(w) - \lambda a w &= 0 \\ L_w = n f_w - \lambda a n &= 0 \\ L_\lambda = B - anw - c &= 0 \end{aligned}$$

Note that second FOC implies that $f_w - \lambda a = 0$.

(5) (c) Construct Jacobian matrix and find its determinant. [Hint: you may want to use the information about $(f_w - \lambda a)$ from previous part.] Given the assumed restrictions on parameters and the properties of the function $f(w)$, what is the sign of this determinant?

Solution: $J = \begin{pmatrix} 0 & f_w - \lambda a & -aw \\ f_w - \lambda a & n f_{ww} & -an \\ -aw & -an & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -aw \\ 0 & n f_{ww} & -an \\ -aw & -an & 0 \end{pmatrix}$.

Note that we utilized information from part (b) that $f_w - \lambda a = 0$.

(5) (d) Use Cramer's rule to find the general comparative static derivatives: $\partial \lambda^*/\partial a$, $\partial n^*/\partial a$, and $\partial w^*/\partial a$. Be sure to indicate the signs of these expressions. Can you give an intuitive explanation of your result for $\partial w^*/\partial a$?

Solution:

$$\frac{\partial w^*}{a} = \frac{\begin{vmatrix} 0 & \lambda w & -aw \\ 0 & \lambda n & -an \\ -aw & nw & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & -aw \\ 0 & n f_{ww} & -an \\ -aw & -an & 0 \end{vmatrix}} = \frac{0}{-(aw)^2 n f_{ww}} = 0$$

It may be surprising that the width of the lane does not depend on the variable cost. The idea is that we want to find an optimum lane width (w^*) that maximizes the flow and then replicate it building more lanes (n^*).

$$\frac{\partial \lambda^*}{a} = \frac{\begin{vmatrix} 0 & 0 & \lambda w \\ 0 & n f_{ww} & \lambda n \\ -aw & -an & nw \end{vmatrix}}{\begin{vmatrix} 0 & 0 & -aw \\ 0 & n f_{ww} & -an \\ -aw & -an & 0 \end{vmatrix}} = \frac{a \lambda w^2 n f_{ww}}{-(aw)^2 n f_{ww}} = -\frac{\lambda}{a} = \frac{-f_w}{a^2} < 0$$

Question 2 continues on the next page...

$$\frac{\partial n^*}{a} = \frac{\begin{vmatrix} \lambda w & 0 & -aw \\ \lambda n & nf_{ww} & -an \\ nw & -an & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & -aw \\ 0 & nf_{ww} & -an \\ -aw & -an & 0 \end{vmatrix}} = \frac{\lambda w(an)^2 - aw(-\lambda an^2 - n^2 w f_{ww})}{-(aw)^2 n f_{ww}} = -\frac{n}{a} < 0$$

3. (25 min) A particular firm uses labor (L) to produce a single output (Q). The marginal product function is given by: $M(L) = \frac{dQ}{dL} = \frac{\alpha A}{L^{1-\alpha}}$, where $0 < \alpha < 1$.

(5) (a) Use integration to find the firm's production function, $Q(L)$. [Assume that the constant of integration is zero.]

Solution:

$$Y(L) = \int \alpha A L^{\alpha-1} dL = AL^{\alpha}$$

(5) (b) If the firm is a profit-maximizer, it will use labor up to the point where the value of marginal product [output price (p) times the marginal product] equals the wage rate (w). Solve this condition to find the expression for the optimal amount of labor (L^*).

Solution:

$$pMP_L = w \Rightarrow p\alpha A L^{\alpha-1} = w \Rightarrow L^* = \left(\frac{w}{p\alpha A}\right)^{1/(1-\alpha)}$$

(5) (c) The firm's maximum profit (Π^*) is given by the area below the value of marginal product curve, $pM(L)$, but above the wage rate, from $L = 0$ to $L = L^*$. Use definite integration and your answer in part (b) to find the expression for Π^*.

Solution:

$$\Pi^* = \int_0^{L^*} pMP_L dL - wL^* =$$

$$= pAL^{\alpha}\Big|_0^{L^*} - wL^*$$

$$\Pi^* = pA(L^*)^{\alpha} - wL^* = pA\left(\frac{p}{w\alpha A}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{p}{w\alpha A}\right)^{\frac{1}{1-\alpha}} = (1-\alpha)\left[pA\left(\frac{\alpha}{w}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}}$$

4. (20 min) As an assistant of the CFO of the company you are working at you have been asked to find a way to decrease the costs of the company. Your company produces paper products and uses wood (W) and glue (G) for production. Based on the amount of wood an glue used in the production, the total output of the company is given by a production function $Y = W \cdot G$. Your choices are limited though. The owners of the company want to be able to produce not less than 100 units of output. Another limiting factor is the availability of glue. Your company uses a specific type of glue and the maximum amount your supplier can provide is M. The price of wood is p_w and the price of glue is p_g

   You recall the Econ 214 that you took 10 years ago and figured out that you need to solve a cost minimization problem. Since you do not recall how to solve minimization problems, you assumed (correctly!) that minimizing a function is the same thing as maximizing the negative of it:

$$\max_{W,G} -(Wp_w + Gp_g)$$

$$\begin{aligned} \text{s.t.} \quad & 100 \leq W \cdot G \\ \text{and} \quad & G \leq M \end{aligned}$$

   Note, that we have two constraints, thus we have to introduce two auxiliary variables λ_1 and λ_2. The Lagrangian for this problem is:

$$L = -Wp_w - Gp_g + \lambda_1[W \cdot G - 100] + \lambda_2(M - G)$$

(5) (a) Write the first order (Kuhn-Tucker) conditions for this problem. (Hint: There are 8 of them.)

> **Solution:** Kuhn-Tucker conditions:
> $L_W = -p_w + \lambda_1 G = 0$
> $L_G = -p_g + \lambda_1 W - \lambda_2 = 0$
> $100 \leq WG$
> $G \leq M$
> $\lambda_1 \geq 0$
> $\lambda_2 \geq 0$
> $\lambda_1[W \cdot G - 100] = 0$
> $\lambda_2(M - G) = 0$

(10) (b) Given the location of the curves in Figure 1 shade the combinations of W and G that are feasible.

   [Hint: The currently shaded area on the graph above indicates combinations of W and G that produce *at least* 100 units.]

(5) (c) Given the feasible set of W and G that you have identified and the position of the curves, try to find the solution for W^* and G^* using the Kuhn-Tucker condition of

Question 4 continues on the next page...

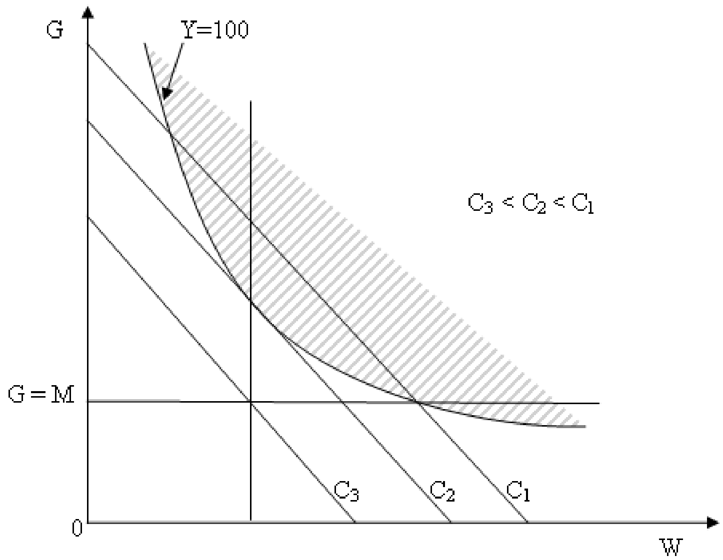

Figure 1: Identify feasible choices

part (a). [Hint: Look at the graph and figure out if any of the constraints (or all) are binding.]

[Superhint: We want to be on the lowest possible cost line (the straight one), but the owners want to have output of not lower than 100]

Solution: Since it is evident from the graph that both constraints are binding, both constraints are satisfied with equality. Thus it follows from the second constraint that $G^* = M$. Plugging this into the first constraint we obtain that $W^* = 100/M$.

5. (15 min) You are a perfectionist. You want to graduate with a perfect GPA if possible. Thus, you don't care if you have to repeat all courses. Your GPA for the year (g_t) is a function of the previous year GPA (g_{t-1}) and evolves according to this difference equation:

$$g_t = 0.5g_{t-1} + 1.5$$

(5) (a) Let's say that first year in college ($t = 1$) your GPA is 2. What will be your GPA in the 4th year?

> **Solution:** $g_1 = 2$
> $g_2 = 0.5g_1 + 1.5 = 2.5$
> $g_3 = 0.5g_2 + 1.5 = 2.75$
> $g_4 = 0.5g_3 + 1.5 = 2.875$

(5) (b) What is the level of GPA that you will achieve if you stay in college indefinitely? [Hint: Find the steady state of the sequence]

> **Solution:** $g_s = 0.5g_s + 1.5 \Rightarrow g_s = 3$

(5) (c) Would your answer to (b) change if it happens to be your GPA in the first year is 3.5?

> **Solution:** No, the answer will not change. GPA will go down to 3 anyway.

Grade table

Question:	1	2	3	4	5	Total
Points:	25	25	15	20	15	100
Score:						