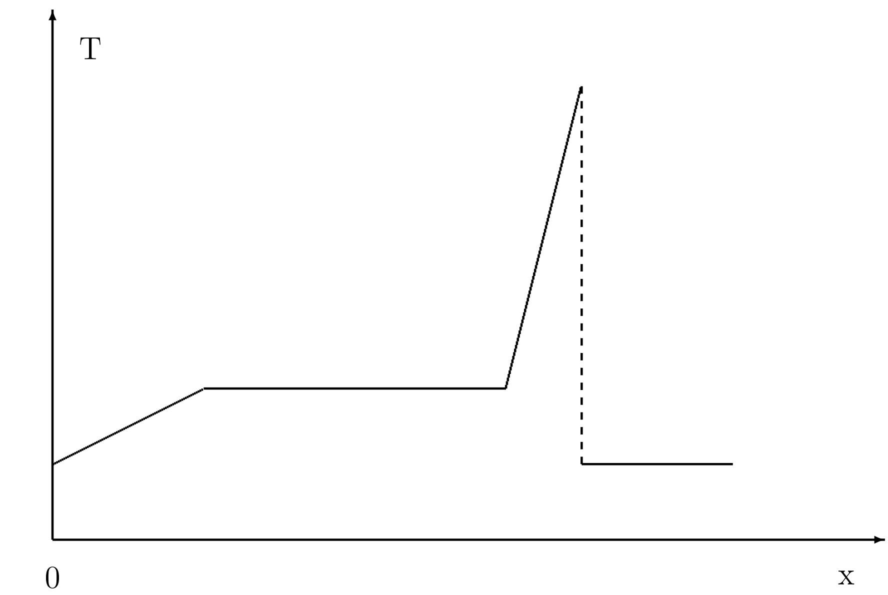
Answer the questions fully to your best ability. Use the space provided. If you run out of room, use the backsides. No partial credit will be given if you do not show the steps of your calculations! **Write as neatly as possible!**

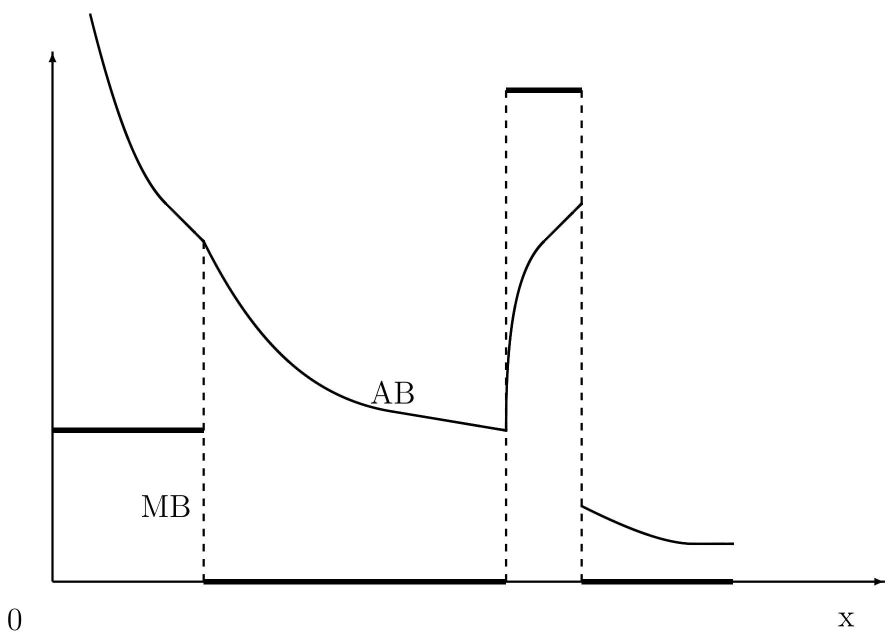
Name: ______________________________

1. The following graph shows how your grade for this course depends on the time you devote to study. Thus we can treat this graph as "Total usefulness of studying" (T). Use the blank diagram to plot the graphs that would show:

(5) (a) The marginal benefit of studying given number of hours.

(5) (b) The average benefit of studying a given number of hours.





(2) 2. Evaluate the following expression: $\lim_{x\to 0} \frac{x^3-2x-c}{x^4} =$

> **Solution:** $\lim_{x\to 0} \frac{x^3-2x-c}{x^4} = \lim_{x\to 0} \frac{\frac{x^3}{x^4}-\frac{2x}{x^4}-\frac{c}{x^4}}{\frac{x^4}{x^4}} = \lim_{x\to 0} \frac{1}{x} - \frac{2}{x^3} - \frac{c}{x^4} = -\infty$
> (Assuming $c > 0$)

3. Are these functions continuous?

(2) (a) $y(x) = \begin{cases} 2-2x & \text{for} \quad x \leq 5 \\ 2x-4 & \text{for} \quad x > 5 \end{cases}$

> **Solution:** No. This function is not continuous. (Try plotting it!)

(2) (b) $y(x) = \frac{1}{x}$

> **Solution:** No. This function is not continuous. A function $f(x)$ is continuous at $x - \bar{x}$ if the right hand and left hand limits are defined and equal. It also must be true that $f(\bar{x}) = \lim_{x\to\bar{x}^+} f(x) = \lim_{x\to\bar{x}^-} f(x)$. This is not the case with $f(x) = 1/x$. Left hand and right hand limits are different as $x \to 0$.

4. Simplify the following expressions:

(2) (a) $\frac{x+1}{x+2} \cdot \frac{2x^2+7x+3}{x^2+2x+1} \div \frac{2x+1}{x+1}$

> **Solution:** $\frac{x+1}{x+2} \cdot \frac{2x^2+7x+3}{x^2+2x+1} \div \frac{2x+1}{x+1} = \frac{(x+1)\cdot(x+1)}{(x+2)\cdot(2x+1)} \cdot \frac{2x^2+7x+3}{x^2+2x+1} = \frac{2x^2+7x+3}{(x+2)\cdot(2x+1)} = \frac{2x^2+7x+3}{2x^2+5x+2} =$
> $= \frac{(2x+1)\cdot(x+3)}{(2x+1)\cdot(x+2)} = \frac{x+3}{x+2}$

(2) (b) $((x^{1/3})^8)^{1/2} \cdot x^2 \div x^{3/4}$

> **Solution:** $((x^{1/3})^8)^{1/2} \cdot x^2 \div x^{3/4} = x^{\frac{1}{3}\cdot 8 \cdot \frac{1}{2} + 2 - \frac{3}{4}} = x^{31/12}$

(2) 5. Find the roots of the function $y = x^2 - 1$

> **Solution:** We need to find solution to $y(x) = 0$:
> Notice that $y = x^2-1 \Rightarrow y = x^2-1+2x-2x \Rightarrow y = x(x-1)+1\cdot(x-1) = (x+1)(x-1)$.
> Thus $y(x) = 0 \to x = 1$ or $x = -1$.

6. Consider the matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 0 \end{pmatrix}, B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 6 continues on the next page…

(2) (a) Is $A+I$ defined? If so, calculate $A+I$. If undefined, explain why.

> **Solution:** No. $A+I$ is undefined, since A and I are of different dimensions.

(2) (b) Is $A \cdot B$ defined? If so, calculate $A \cdot B$. If undefined, explain why.

> **Solution:** No. $A \cdot B$ is not defined, since matrix multiplication AB is not conformable. A is 2×3 matrix and B is 2×2 matrix.

(2) (c) Is $B \cdot A$ defined? If so, calculate $B \cdot A$. If undefined, explain why.

> **Solution:** Yes, $B \cdot A$ is defined. $B \cdot A = \begin{pmatrix} a+4b & 2a+3b & 3a \\ c+4d & 2c+3d & 3c \end{pmatrix}$

(2) (d) Is $A \cdot I$ defined? If so, calculate $A \cdot I$. If undefined, explain why.

> **Solution:** Yes. $A \cdot I$ is defined. $A \cdot I = A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 0 \end{pmatrix}$

(2) (e) Is $A \cdot A'$ defined? If so, calculate $A \cdot A'$. If undefined, explain why.

> **Solution:** Yes. $A \cdot A'$ is defined. $A \cdot A' = \begin{pmatrix} 14 & 10 \\ 10 & 25 \end{pmatrix}$

(2) (f) Is the determinant of I defined? If so, calculate the determinant of I. If undefined, explain why.

> **Solution:** Yes. Determinant of I is defined. It is 1. (The multiplication of the diagonal elements.)

(2) (g) Is $A'B - B$ defined? If so, calculate $A'B - B$. If undefined, explain why.

> **Solution:** No. $A'B - B$ is undefined. A' is 3×2 matrix thus $A'B$ is 3×2 matrix. B is 2×2 matrix, thus $A'B - B$ is not conformable.

(2) (h) Is $(A'B - B)^{-1}$ defined? If so, calculate $(A'B - B)^{-1}$. If undefined, explain why.

> **Solution:** No. $(A'B - B)^{-1}$ is not defined. Since $A'B - B$ is not defined (part g), it's inverse is not defined also.

(2) (i) Assume λ is a scalar. Is the determinant of λI defined? If so, calculate determinant of λI. If undefined, explain why.

Solution: $\lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ Thus $|\lambda I| = \lambda^3$.

(2) 7. What are the dimensions of matrix M, if $M = (X' \cdot Y)^{-1}$, and X is 5×2 matrix and Y is 5×2 matrix? Maybe $(X' \cdot Y)^{-1}$ is undefined?

Solution: M is 2×2. X' is 2×5, thus $X'Y$ is 2×2. We can not tell if $(X'Y)^{-1}$ exists or not. $X'Y$ is a square matrix, but that is not sufficient. If an inverse exists, it is a 2×2 matrix.

8. Consider the matrix $H = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}$

(2) (a) Find the determinant of H.

Solution: Expanding along the 3rd column:
$|H| = 0 \cdot |somthing| - 1 \cdot \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + 0 \cdot |somthing| = -10$

(2) (b) Find the cofactor matrix C for matrix H.

Solution: Cofactor matrix is a signed minor matrix:
$C = \begin{pmatrix} -3 & 2 & -3 \\ 0 & 0 & -10 \\ 1 & -4 & 1 \end{pmatrix}$

(2) (c) Find the adjoint matrix, $adj(H)$ for matrix H.

Solution: Adjoint of H, $adj(H)$, is the transposed cofactor matrix.
$adj(H) = C' = \begin{pmatrix} -3 & 0 & 1 \\ 2 & 0 & -4 \\ -3 & -10 & 1 \end{pmatrix}$

(2) (d) Find H^{-1}, the inverse of H.

Solution: $H^{-1} = \frac{1}{|H|} \cdot adj(H) = -\frac{1}{10} \cdot \begin{pmatrix} -3 & 0 & 1 \\ 2 & 0 & -4 \\ -3 & -10 & 1 \end{pmatrix} = \begin{pmatrix} 3/10 & 0 & -1/10 \\ -1/5 & 0 & 2/5 \\ 3/10 & 1 & -1/10 \end{pmatrix}$

(1) (e) Verify that the solution to above is correct.

Solution: $H \cdot H^{-1} = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3/10 & 0 & -1/10 \\ -1/5 & 0 & 2/5 \\ 3/10 & 1 & -1/10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9. Consider the export/import model:

$$\begin{aligned} X &= 1000 - 20E + 0.2Y_F \\ M &= 450 + 10E + 0.15Y_D \\ X &= M \end{aligned}$$

where X represents exports, M represents imports, E represents the exchange rate, Y_F represents foreign income, and Y_D represents domestic income. The exogenous variables include Y_D and Y_F, and endogenous variables are X, M, and E.

(4) (a) Rewrite this system of equations in matrix form $Ax = d$. [Note: You can eliminate X or M from the original system of equations, reducing it to **two equations**.]

Solution: Utilizing $X = M$ we can rewrite the system as:

$$\begin{aligned} M &= 1000 - 20E + 0.2Y_F \\ M &= 450 + 10E + 0.15Y_D \end{aligned}$$

Which can be written in matrix form $Ax = d$:

$$\begin{pmatrix} 1 & 20 \\ 1 & -10 \end{pmatrix} \cdot \begin{pmatrix} M \\ E \end{pmatrix} = \begin{pmatrix} 1000 + 0.2Y_F \\ 450 + 0.15Y_D \end{pmatrix}$$

(4) (b) Find the determinant of A.

Solution: $|A| = 1 \cdot (-10) - 1 \cdot 20 = -30$

(6) (c) Find the solution to this system of equations. [Hint: You can do it by substitution, or using matrix algebra $x = A^{-1}d$.]

Solution: Using $x = A^{-1}d \Rightarrow x = -1/30 \cdot \begin{pmatrix} -10 & -20 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1000 + 0.2Y_F \\ 450 + 0.15Y_D \end{pmatrix}$

$$\begin{pmatrix} M \\ E \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 1/30 & -1/30 \end{pmatrix} \cdot \begin{pmatrix} 1000 + 0.2Y_F \\ 450 + 0.15Y_D \end{pmatrix} = \begin{pmatrix} \frac{1900}{3} + \frac{1}{15}Y_F + \frac{1}{10}Y_D \\ \frac{55}{3} + \frac{1}{150}Y_F - \frac{1}{200}Y_D \end{pmatrix}$$

Remember that $X = M = \frac{1900}{3} + \frac{1}{15}Y_F + \frac{1}{10}Y_D$.

Solution by substitution:
$1000 - 20E + 0.2Y_F = 450 + 10E + 0.15Y_D$
$30E = 550 + 0.2Y_F - 0.15Y_D$
$E^* = \frac{55}{3} + \frac{1}{150}Y_F - \frac{1}{200}Y_D$
$M^* = 450 + 10 \cdot (\frac{55}{3} + \frac{1}{150}Y_F - \frac{1}{200}Y_D) + 0.15Y_D$
$X^* = M^* = \frac{1900}{3} + \frac{1}{15}Y_F + \frac{1}{10}Y_D$

Grade table

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	2	4	4	2	18	2	9	14	65
Score:										