| Answer the questions fully to your best ability. Use the space provided. If you run out of room, use the backsides. No partial credit will be given if you do not show the steps of your calculations! **Write as neatly as possible!** |
|---|

Name: ____________________

1. Consumer derives utility of the consumption of two goods - apples and bananas. The specific utility function is of this form:

$$U(a, b) = Ca^{\alpha}b^{\beta}$$

Where C is some positive constant, $a$ is the amount of apples, and $b$ is the amount of bananas consumed. $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta = 1$.

(2) (a) What is the marginal utility of apples ($MU_a$) [1]?

> **Solution:** $MU_a = \alpha Ca^{\alpha-1}b^{\beta}$

(2) (b) Does this utility function exhibit diminishing marginal utility of apples? Explain.

> **Solution:** Yes. $\frac{\partial MU_a}{\partial a} = \alpha(\alpha - 1)Ca^{\alpha-2}b^{\beta} < 0$

(2) (c) What is the marginal utility of bananas($MU_b$)?

> **Solution:** $MU_b = \beta Ca^{\alpha}b^{\beta-1}$

(2) (d) Does this utility function exhibit diminishing marginal utility of bananas? Explain.

> **Solution:** Yes. $\beta(\beta - 1)Ca^{\alpha}b^{\beta-2} < 0$

(2) (e) Is $U(a, b)$ homogeneous? If so of what degree?

> **Solution:** Yes. $U(sa, sb) = s^{\alpha+\beta}U(a, b)$, and $\alpha + \beta = 1$. Thus homogeneous of degree 1.

(2) (f) Is $MU_a$ (marginal utility of apples) homogeneous? If so of what degree?

> **Solution:** Yes. $MU_a(sa, sb) = s^{\alpha-1+\beta}MU_a$. Thus homogeneous of degree 0.

2. Find the stationary points of these functions, identify if it is a maximum, a minimum or an inflection point:

[1] Notice, that $MU_a$ does not mean a derivative of the marginal utility with respect to a, but simply marginal utility of apples.

Question 1 continues on the next page…

(2) (a) $y = x^2 - 2x + 1$

**Solution:** $y' = 2x - 2 = 0 \rightarrow x^* = 1$
$y''(x) = 2$, thus a function is convex and at $x^* = 1$ the function attains a global minimum.

(2) (b) $y = x$
[Hint: Try plotting this function]

**Solution:** There is no stationary point for this function.

(2) (c) $y = x^4 - x^3$
[Hint: This function has two stationary points]

**Solution:** $y' = 4x^3 - 3x^2 = 0 \rightarrow x^2(4x - 3) = 0 \rightarrow x_1^* = 0$ and $x_2^* = 3/4$.
Checking second order condition:
$y'' = 12x^2 - 6x$
At $x_1^* : y''(0) = 0 \rightarrow$ Check using the n$^{th}$ derivative test. $y^{(3)}(x) = 24x - 6$. $y^{(2)}(0) = 0, y^{(3)}(0) = -6$, thus the first nonnegative derivative for $x_1^*$ is the 3$^{rd}$ derivative (odd), thus at $x_1^*$ y(x) has an inflection point.
At $x_2^* : y''(3/4) = 9/4 > 0$, thus at $x_2^*$ a function attains a local minimum.

3. Find the first-order and second-order derivatives of the following function:

$$z = \frac{e^y}{\ln(x)}$$

(2) (a) $z_x$

**Solution:** $-\frac{e^y}{x \ln(x)^2}$

(2) (b) $z_{xx}$

**Solution:** $\frac{e^y (2+\ln(x))}{x^2 \ln(x)^3}$

(2) (c) $z_{xy}$

**Solution:** $-\frac{e^y}{x \ln(x)^2}$

(2) (d) $z_y$

**Solution:** $\frac{e^y}{\ln(x)}$

(2) (e) $z_{yy}$

**Solution:** $\frac{e^y}{\ln(x)}$

(2) (f) $z_{yx}$

**Solution:** $-\frac{e^y}{x \ln(x)^2}$

4. You want to allocate your time to two classes that you are taking. The total time that you want to devote for studying is $T$ hours. The grade that you are going to get in each of the class is a function of the time spent studying: $f(t_1, t_2)$ for the first class and $g(t_2)$ for the second (measured in grade points). You want your GPA for the two classes to be equal to $G$. We can write the above conditions in a system of equations:

$$\begin{cases} t_1 + t_2 - T = 0 \\ f(t_1, t_2) + g(t_2) - 2G = 0 \end{cases}$$

(2) (a) Find the Jacobian of the given system of equations.

**Solution:** $J = \begin{pmatrix} 1 & 1 \\ f_1 & f_2 + g_2 \end{pmatrix}$

(2) (b) Does a unique solution to this system of equations exist? What does your answer depend on?

**Solution:** $|J| = f_2 + g_2 - f_1$ If $|J| = 0$ there is no unique solution.

(2) (c) Use the implicit function theorem approach to find expressions for the comparative static derivatives $\frac{\partial t_1^*}{\partial G}, \frac{\partial t_2^*}{\partial G}$. Do not try to evaluate the sign of these expressions!

**Solution:** $\frac{\partial t_1^*}{\partial G} = \dfrac{\begin{vmatrix} 0 & 1 \\ 2 & f_2 + g_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ f_1 & f_2 + g_2 \end{vmatrix}} = -2(f_2 + g_2 - f_1)$

$$\frac{\partial t_2^*}{\partial G} = \dfrac{\begin{vmatrix} 1 & 0 \\ f_1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ f_1 & f_2 + g_2 \end{vmatrix}} = 2(f_2 + g_2 - f_1)$$

5. A monopolist cellular service provider "Big Ear" is faced with the following demand for it's service: $p = f(q, m)$, where $p$ is the price for the monthly contract, $q$ is the amount of monthly service contracts signed, and $m$ is the amount of air time minutes included in the contract. To have some control over the company, government regulates how many minutes are included in the contract. Thus the company has no choice but just to comply with this regulation.

   "Big Ear" incurs a cost of $t$ per one contract signed and tries to maximize it's profit, by choosing the number of contracts to be sold:

$$\max_q f(q, m) \cdot q - t \cdot q$$

   It is assumed, that $f_q < 0, f_{qq} = 0, f_{qm} = 0, f_m > 0$.

(4) (a) Find the first order condition for profit maximization.

**Solution:** $\frac{\partial \Pi}{\partial q} = f_q(q, m) \cdot q + f(q, m) - t = 0$

(2) (b) Verify that the second order condition for maximum is satisfied.

**Solution:** $\frac{\partial^2 \Pi}{\partial q^2} = q \cdot f_{qq} + f_q + f_q = 2f_q + q \cdot f_{qq} < 0$. This implies that the profit function is concave, thus the SOC for a maximum is satisfied.

(2) (c) Government wants to know, what will happen to the number of the contracts sold, if they change the regulation increasing the number of minutes $m$ included in the monthly contract. Apply the implicit function rule to find the comparative static derivative $\frac{\partial q*}{\partial m}$. What is the sign of this derivative? Does it make economic sense?

**Solution:** $F(q, m, t) = f_q(q, m) \cdot q + f(q, m) - t = 0$
$\frac{\partial q*}{\partial m} = -\frac{F_m}{F_q} = -\frac{f_m + q \cdot f_{qm}}{f_{qq} \cdot q + f_q + f_q} = -\frac{f_m}{2f_q} > 0$
Yes, it does make economic sense. Since the increase in the minutes included in the contract is going to increase demand and does not cause any increase in the cost, it is expected that more contracts will be signed.

(2) (d) We also would like to know, what will happen to the number of the contracts signed if the marginal cost $t$ increases. Apply the implicit function rule to find the comparative static derivative $\frac{\partial q*}{\partial t}$. What is the sign of this derivative? Does it make economic sense?

**Solution:** $F(q, m, t) = f_q(q, m) \cdot q + f(q, m) - t = 0$
$\frac{\partial q*}{\partial t} = -\frac{F_t}{F_q} = -\frac{-1}{f_{qq} \cdot q + f_q + f_q} = \frac{1}{2f_q} < 0$
Yes, it does make economic sense. "Big Ear" is experiencing an increase in the marginal cost, and no increase in the demand, thus it is expected that the equilibrium amount of the contracts signed will go down.

6. Find the total differential of the following functions:

(2) (a) $y = 2e^{x_1x_2} + \ln(x_1x_2)$

> **Solution:** $dy = [2e^{x_1x_2} \cdot x_2 + \frac{1}{x_1x_2} \cdot x_2] \cdot dx_1 + [2e^{x_1x_2} \cdot x_1 + \frac{1}{x_1x_2} \cdot x_1] \cdot dx_2$
> $dy = [2x_2e^{x_1x_2} + \frac{1}{x_1}] \cdot dx_1 + [2x_1e^{x_1x_2} + \frac{1}{x_2}] \cdot dx_2$

(2) (b) $z = \left(\frac{2x^2}{\ln(y)}\right)^2$

> **Solution:** $dz = \frac{16x^3}{(\ln(y))^2} \cdot dx - \frac{8x^4}{y(\ln(y))^3} \cdot dy$

(0) 7. [Extra credit] Find the derivative of the following function: $y = x^x$

> **Solution:** Take the log of both sides: $\ln(y) = x\ln(x)$. Rearrange to get an implicit function: $F = \ln(y) - x\ln(x) = 0$.
>
> Apply the implicit function theorem:
> $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\ln(x)-x\cdot\frac{1}{x}}{\frac{1}{y}} = y(\ln(x)+1) = x^x(\ln(x)+1)$.

**Grade table**

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| Points: | 12 | 6 | 12 | 6 | 10 | 4 | 0 | 50 |
| Score: | | | | | | | | |